Name: ____

Student ID: ____

Honor Code: As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

Signature: ____

This is a closed-book, closed-calculator, etc. exam. Remember to carefully justify every statement that you write, and to follow the style of proper mathematical writing. Unless otherwise indicated, you can use results proved in lecture, the textbook, and homework, provided they are clearly stated. If necessary, continue solutions on backs of pages.

The exam is out of 60 points, but there are 5 points of bonus credit available, for a maximum possible score of 65.

Time limit: 110 minutes.

Problem	Total Points	Earned Points
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7 (Bonus)	Up to 5	
Total	60	

1. (**10 points**) Let $\mathbf{R}$ be the set of real numbers, and let $\sim$ be the equivalence relation given by $x \sim y$ if and only if $x - y \in \mathbf{Q}$ (you do not need to prove that this is an equivalence relation). Determine, with proof, if the quotient set $\mathbf{R}/\sim$ is countable or uncountable.

2. (**10 points**) Suppose we select a set S of 13 different integers. Show that there are two different integers $n, m \in S$ such that $n^2 \equiv m^2 \bmod 35$. [Hint: pigeonhole principle.]

3. (**10 points**) Suppose x is a real number with $x + 1/x \in \mathbf{Q}$. Show that for all $n \geq 1$, $x^n + 1/x^n \in \mathbf{Q}$.

4. (**10 points**) Let $n \geq 3$ be an integer. Show that

$$\sum_{i=2}^{n-1} (i-1)(n-i) = \binom{n}{3}.$$

5. (**10 points**) Let X and Y be independent random variables such that X takes on the values $\{0, 1, 2\}$ with the uniform distribution, and Y takes on the values 0 and 1 with probabilities 1/3 and 2/3, respectively. Let Z be the random variable whose output is the unique integer in $\{0, 1, 2\}$ that is equivalent to $X + Y$ mod 3. Show that

$$E(XZ) = E(X)E(Z),$$

but X and Z are not independent.

6. (**10 points**) Let G be a (finite, simple) bipartite graph with vertex bipartition $\{V_1, V_2\}$. Suppose that all vertices of G have the same degree $n \geq 1$. Show that $|V_1| = |V_2|$.

7. (**Bonus Problem, 5 points**) Prove that if p is an odd prime, then

$$\left(\frac{2}{p}\right) \equiv (-1)^{(p^2-1)/8} \bmod p.$$

This is known as the *second supplement to quadratic reciprocity*. [Hint: show that $2^s s! \equiv s!(-1)^{s(s+1)/2} \bmod p$, where $s = (p-1)/2$.]